



One Knob, Infinitely Many Histories



COMPLETE TO COLLECT
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 Level 2 · 4 Glints

TURN ONE KNOB; BEND A WHOLE HISTORY

A puck starts at position 0 at time 0 and ends at position 1 at time 1. Turning the parameter creates infinitely many histories with these same endpoints.

$$x_a(t) = t + a t(1 - t).$$

Changing a changes the entire motion while keeping its endpoints fixed.

THE SYMBOLS PLAY TWO DIFFERENT ROLES

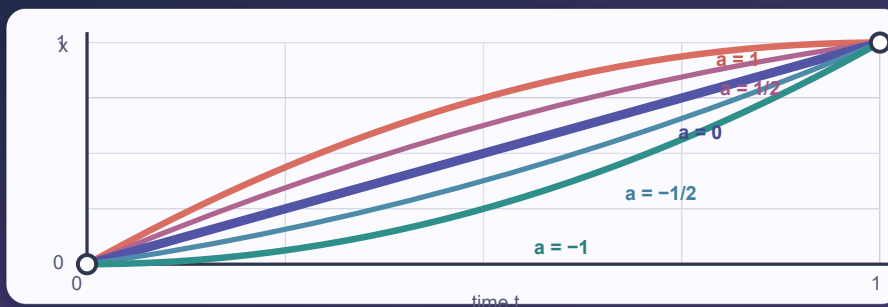
Time t moves along one curve

Hold a fixed. The derivative $x'_a(t)$ is that history's velocity.

The knob a selects a curve

Hold the endpoints fixed. Changing a changes the whole history at once.

First study the family and its velocities. On page 2, you will assign one action number to each curve and then differentiate with respect to a .



HISTORY KNOB



rush early
 late early

The display shows five settings; the formula defines a history for every real value of a .

PAGE 1 · UNDERSTAND THE FAMILY Show derivatives, integrals, and algebra on separate paper.

- Verify that every history in the family has the required endpoints. Differentiate to find its velocity. Which sign of the parameter makes the puck rush early, and which makes it rush late?



ACTION OF ONE HISTORY

$$S(a) = \int_0^1 \left(x'_a(t) \right)^2 dt$$

After integration, S is a function of the knob setting. For this family, **stationary action** means $S'(a) = 0$.

- Substitute the velocity into the action. Expand the square, use the three supplied integrals, and show that infinitely many histories reduce to one action function.

$$\int_0^1 1 dt = 1 \qquad \int_0^1 (1 - 2t) dt = 0 \qquad \int_0^1 (1 - 2t)^2 dt = \frac{1}{3}$$

Action function $S(a) =$ _____

- Differentiate the action function with respect to the knob setting. Find the stationary parameter, identify the corresponding history, and use the action formula to decide whether it is a minimum.

$$S'(a) =$$

$$\text{stationary } a =$$

$$x(t) =$$

- Explain why opposite knob settings have equal action even though one rushes early and the other rushes late.

Next idea: Calculus of variations tests arbitrary endpoint-fixing changes $\varepsilon\eta(t)$, not only this one knob.

