



How Far East Can the Mandelbrot Set Go?



COMPLETE TO COLLECT

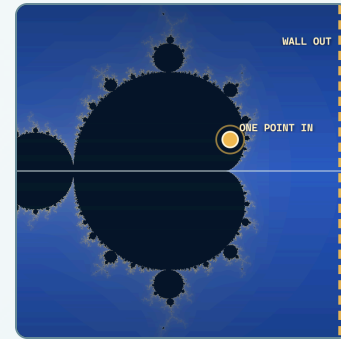
The Mandelbrot Marquise

Unique Jewel · 6 Glints

THE FULL COMPLEX SET

Let M be the Mandelbrot set: the complex values c for which the orbit $z_0 = 0$, $z_{n+1} = z_n^2 + c$ stays bounded. If any $|z_n| > 2$, the orbit escapes.

Write $c = x + yi$. Its horizontal coordinate is $\text{Re}(c) = x$. Let R be the greatest real part of any point in M . Your goal is to build a rigorous **lower and upper bound** for R .



One point in gives a floor. One whole region out gives a ceiling.

PRODUCTS MULTIPLY DISTANCES

$$|uv| = |u| |v|$$

A DETOUR CANNOT BE SHORTER

$$|u + v| \leq |u| + |v|$$

CERTIFICATE A · PUT ONE POINT INSIDE

A safe disk will control infinitely many iterates.

- **Create a point to test.** Let

$$a = \frac{1}{5} + \frac{\sqrt{3}}{5}i, \quad c_0 = a - a^2.$$

Compute a^2 and c_0 . Then record $\text{Re}(c_0)$.

- **Trap its orbit.** Let D be the disk $|z - a| \leq \frac{1}{5}$.

1. Since $z_1 = c_0$, prove $|z_1 - a| = |-a^2| < \frac{1}{5}$.

2. Use $c_0 = a - a^2$ to factor:

$$z_{n+1} - a = z_n^2 + c_0 - a =$$

3. Assume $|z_n - a| \leq \frac{1}{5}$. Prove $|z_{n+1} - a| \leq \frac{1}{5}$, then use your factorization to prove $|z_{n+1} - a| \leq \frac{1}{5}$.



- **Build an eastern wall.** The second iterate is $z_2 = c^2 + c$. Expand its real and imaginary parts, then verify

$$|z_2|^2 = (x^2 + x)^2 + (2x^2 + 2x + 1)y^2 + y^4.$$

Every omitted term on the right is nonnegative. Explain why $x > 1$ forces $|z_2| > 2$. What upper bound does this prove for R ?

- **Close the cage.** Combine your two certificates into one inequality for R . Then explain the different logical jobs:
 - Why is *one* bounded parameter enough for the lower bound?
 - Why must the upper-bound proof exclude *every* parameter beyond the wall?

$$\underline{\hspace{2cm}} \leq R \leq \underline{\hspace{2cm}}$$

Lower bound: exhibit one point in M

Upper bound: exclude every point past a wall

