



The Last Safe Positive Real Number



COMPLETE TO COLLECT

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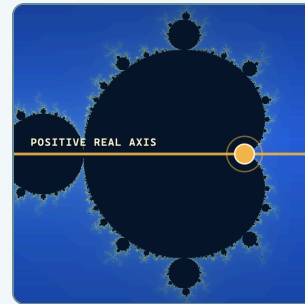
Level 3 · 5 Glints

A COMPLEX SET, ONE REAL SLICE

The full **Mandelbrot set** is built by choosing a complex number c , starting at $z_0 = 0$, and repeating

$$z_{n+1} = z_n^2 + c.$$

If some iterate has $|z_n| > 2$, the orbit escapes and c is outside the set. This problem studies only **positive real** values of c . Then every z_n is real, so all the work below uses ordinary real-number algebra.



The full set reaches farther right off this line. Here you will prove only the positive real endpoint.

WHAT IS THE GREATEST POSITIVE REAL c IN THE SET?

Write a complete proof on separate paper.

A greatest-value proof has two jobs: **include the candidate**, then **exclude every larger value**. The two certificates below divide those jobs.

Page 1 · **At the edge:** trap every iterate

Page 2 · **Past the edge:** gain at least d each step

- **Build a safe zone.** Let $c = \frac{1}{4}$. Assume a current value satisfies $0 \leq z_n \leq \frac{1}{2}$. Prove that the next value also satisfies $0 \leq z_{n+1} \leq \frac{1}{2}$. Then use $z_0 = 0$ to explain why this orbit never escapes.



- **PAGE 2 · EXCLUDE EVERY LARGER VALUE**

Move just to the right. Let $c = \frac{1}{4} + d$, where $d > 0$. Rewrite the step increase as a square plus d :

$$z_{n+1} - z_n = z_n^2 - z_n + \frac{1}{4} + d = \underline{\hspace{10em}}$$

Use your rewrite to prove that $z_{n+1} - z_n \geq d$ at every step.

- **Force an escape.** Explain why the guaranteed increase gives $z_n \geq nd$. Then choose a whole-number step n that makes $z_n > 2$.
- **Close both sides.** State the greatest positive real value of c in the Mandelbrot set. Explain how Part 1 proves that value is included and Parts 2–3 rule out every larger positive real value.

