



Two Kinds of One



COMPLETE TO COLLECT

Tumbled Jade

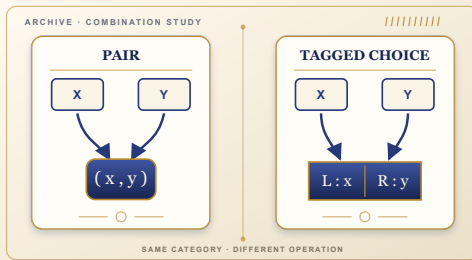
Level 2 · 5 Glints

MEDITATIONS ON ONENESS**ARCHIVE FOLIO** // // // //**ONE WORD, TWO JOBS**

An Archive folio fixes the category **Set** but places two combination instruments beside it. The sets and functions stay fixed; only the chosen operation changes.

*"A unit is not an object alone.
Name the operation."*

— Vivian

**THE TWO DEFINITIONS**

Recall that a set T is **terminal** when every set X has exactly one function $X \rightarrow T$. A bijection is an isomorphism in **Set**.

Now fix a way $\star$ to combine sets. Both combinations below have their standard associative rebracketing and coherence; you may use those facts without proof.

A **unit** for $\star$ is a set I with bijections

$$\lambda_X : I \star X \rightarrow X, \quad \rho_X : X \star I \rightarrow X$$

for every set X , using one rule that respects every function.

Precisely, if $h : X \rightarrow Y$, then **naturality** means

$$h \circ \lambda_X = \lambda_Y \circ (\text{id}_I \star h),$$

$$h \circ \rho_X = \rho_Y \circ (h \star \text{id}_I).$$

For these standard operations, the natural left and right unit maps are the usual monoidal-unit data. On the combined sets, h changes only the X -entry.



TWO COMBINING RULES

Pairs

$X \times Y$ is the set of ordered pairs (x, y) .

Tagged choice

$$X + Y = \{(L, x) : x \in X\} \cup \{(R, y) : y \in Y\}.$$

The tags keep the two sides separate, even if $X = Y$. Here $+$ means tagged union, not arithmetic addition.

1. SEE BOTH RULES

Use separate paper for your lists.

Let $X = \{a, b\}$.

- List the elements of $\{\star\} \times X$ and $X \times \{\star\}$.
- List the elements of $\emptyset + X$ and $X + \emptyset$.
- In each list, describe the evident bijection back to X .

MEDITATIONS ON ONENESS

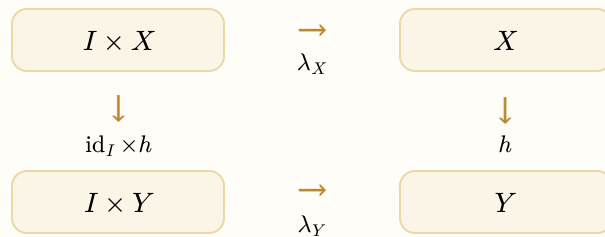
The unit laws

WHAT A COMPLETE UNIT PROOF MUST SHOW

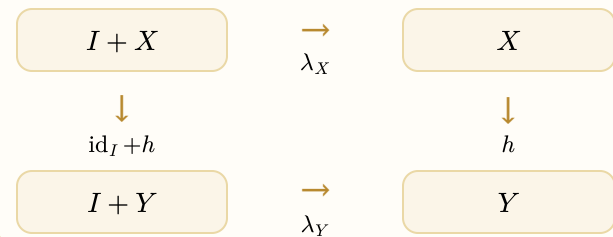
For each combining rule, give formulas for both unit maps and their inverses. Then check that changing the name of an X -element before or after removing the unit has the same result.

NATURALITY DIAGRAMS

Pairs



Tagged choice



2-4. PROVE THE UNIT LAWS

Use separate paper for your formulas and verification.

- For every set X , prove that $\{\star\}$ is a unit for $\times$. Give formulas for λ_X , ρ_X , and both inverses.
- For every set X , prove that $\emptyset$ is a unit for $+$. Give formulas for λ_X , ρ_X , and both inverses.
- Let $h : X \rightarrow Y$. Verify the left naturality equation once for the product unit and once for the tagged-union unit. State why the right equations work the same way.



- Prove both classifications.

- If I is any unit for $\times$, prove that I must have exactly one element. Use the test set $X = \{\star\}$.
- If J is any unit for $+$, prove that J must be empty. Use the test set $X = \emptyset$.

- Complete the comparison table.

Representative set	Terminal in Set?	Unit for $\times$?	Unit for $+$?
$\emptyset$			
$\{\star\}$			

- The arrows of Set stayed the same throughout the problem. Only the combining rule changed. In two precise sentences, explain why **terminal object** and **monoidal unit** are different meanings of one.

