



The Circle That Returns



COMPLETE TO COLLECT

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Level 3 · 6 Glints

MEDITATIONS ON ONENESS · LOCAL FOLIO

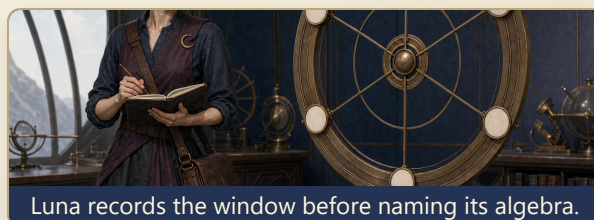
MARK 1111111

LUNA'S SIXFOLD WINDOW

At the Observatory, Luna records a brass window whose six markers are spaced by equal turns. For an angle θ , write

$$R(\theta) = \cos \theta + i \sin \theta.$$

This is the point at angle θ on the unit circle. Angles differing by 360° name the same point.



Luna records the window before naming its algebra.

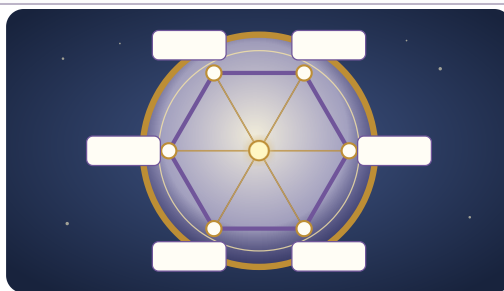
FIELD PLATE · RULES FOR THIS INVESTIGATION

MULTIPLICATION ADDS TURNS

$$R(\alpha)R(\beta) = R(\alpha + \beta)$$

$$[rR(\theta)]^6 = r^6 R(6\theta), \quad r \geq 0$$

Let $\zeta = R(60^\circ)$. Then $\zeta^2 = R(120^\circ)$ and $\zeta^6 = R(360^\circ) = 1$.



Write both the $R(\theta)$ name and the matching power of ζ at every vertex.

FIND EVERY SIXTH ROOT

Label the window here. Use separate paper for the proof.

- Label the six vertices with $R(0^\circ), R(60^\circ), \dots, R(300^\circ)$ and $\zeta^0, \zeta^1, \dots, \zeta^5$. Prove that these six points are **all** the solutions of $z^6 = 1$.

For the “no others” part, write $z = rR(\theta)$. What must $z^6 = 1$ force r and θ to be?



LUNA'S TRACE RECORD · REPEATED MULTIPLICATION WHICH ROOTS VISIT THE WHOLE CIRCLE?

Because $\zeta^6 = 1$, exponents differing by 6 name the same root. For example, $\zeta^8 = \zeta^2$.

Starting at $1 = \zeta^0$, repeatedly multiply by the root in the first column. Record the exponents visited before returning to 0.

Some traces close early. The complete state record will show which ones.

TRACE THE CYCLES

Complete the cycle record on this page.

- A root is a **generator** when its repeated powers visit all six roots.

Complete the table. Which roots are generators?

Multiplier	Exponents visited, beginning with 0	First return after	Visits all six?
ζ^1			
ζ^2			
ζ^3			
ζ^4			
ζ^5			



ARCHIVE SYNTHESIS · ONE OBJECT, ONE OPERATION
WHEN A SET BECOMES A GROUP

A **group** is a set with an operation having these properties:

- multiplying two members stays in the set;
- one identity leaves every member unchanged;
- parentheses do not change the result;
- every member has an inverse that multiplies with it to give the identity.

You may use the fact that complex multiplication is associative.

PROVE THE HIDDEN STRUCTURE

Use separate paper for your group proof.

Prove that the six solutions of $z^6 = 1$ form a group under multiplication.

Use $\zeta^a \zeta^b = \zeta^{a+b}$ and reduce exponents modulo 6. Name the identity and the inverse of every ζ^k . Finally, explain why this group is **cyclic: one root's repeated powers produce every root.**

